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BS-STABILITIES AND ρ-STABILITIES FOR FUNCTIONAL DIFFERENCE EQUATIONS WITH INFINITE DELAY

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Abstract. We study the BS-stability and the ρ -stability for functional difference equations with infinite delay as a discretization of Murakami and Yoshizawa's results [6] for functional differential equation with infinite delay.

1. Introduction

Let $\mathbb{R}^d$ denote the Euclidean d-space with the Euclidean norm $|\cdot|$. Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ denote the set of integers, the set of nonnegative integers, and the set of nonpositive integers, respectively. We denote by $BS(\mathbb{Z},\mathbb{R}^d)$ the set of all bounded functions mapping $\mathbb{Z}$ into $\mathbb{R}^d$ with

$$|\phi|_{BS} = \sup_{s\in\mathbb{Z}} |\phi(s)|.$$

The abstract phase space B is defined as follows [4]: For any function $x : (-\infty, a) \to \mathbb{R}^d$ and $n < a \in \mathbb{Z}$, we define a function $x_n : \mathbb{Z}^- \to \mathbb{R}^d$ by

$$x_n(s) = x(n + s),\ s \in \mathbb{Z}^-.$$

Let B be a real linear space of functions mapping $\mathbb{Z}^-$ into $\mathbb{R}^d$ with a complete seminorm $|\cdot|_B$. We assume the following conditions on the space B.

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(A1) There exist positive constants J, M and N with the property that if $x : (-\infty, a) \to \mathbb{R}^d$ is define on $[\sigma, a)$ with $x_\sigma \in B$ for some $\sigma < a$, $a \in \mathbb{Z}$, then for all $n \in [\sigma, a)$
 (i) $x_n \in B$,
 (ii) $J|x(n)| \leq |x_n|_B \leq M \sup_{\sigma \leq s \leq n} |x(s)| + N|x_\sigma|_B$.

(A2) If (ϕ^k) is a sequence in $B \cap BS$ converging to a function ϕ uniformly on any compact interval in $\mathbb{Z}^-$ and $\sup_k |\phi^k|_{BS} < \infty$, then
 $\phi \in B$ and $|\phi^k - \phi|_B \to 0$ as $k \to \infty$.

The space B contains BS and there is a constant $l > 0$ such that

$$|\phi|_B \leq l|\phi|_{BS}, \phi \in BS. \tag{1.1}$$

The space B is called a *fading memory space* for $\mathbb{Z}$ if it satisfies (A1) and (A2), and the following fading memory condition:

(A3) If $x : \mathbb{Z} \to \mathbb{R}^d$ is a function with $x_0 \in B$, and $x(n) \equiv 0$ on $\mathbb{Z}^+$, then $|x_n|_B \to 0$ as $n \to \infty$.

The space BS which consists of all bounded functions mapping $\mathbb{Z}$ into $\mathbb{R}^d$ is important for the space of initial functions in the theory of functional difference equations with infinite delay [4]. In connection with the stability problems, there are two ways to provide the metric structure in BS. One way is to provide it with the supremum norm, and the other is of compact open topology induced by the ρ-metric. So there are two stability concepts referred to as the BS-stabilities and the ρ-stabilities, respectively. In [6], Murakami and Yoshizawa investigated the relationships between two stabilities in functional differential equations with infinite delay.

In this paper, in order to study the BS-stability and the ρ-stability for the functional difference equation with infinite delay, we will employ to change Murakami and Yoshizawa's results [6] for the functional differential equation with infinite delay into theorems for the functional difference equation.

Consider the functional difference equation

$$x(n+1) = f(n, x_n), n \in \mathbb{Z}^+, \tag{1.2}$$

where $f : \mathbb{Z}^+ \times B \to \mathbb{R}^d$.

For a bounded solution $u(n)$ of (1.2) let K be a compact set in $\mathbb{R}^d$ such that $u(n) \in K$ for all $n \in \mathbb{Z}$, where $u(n) = \phi^0(n)$ for $n < 0$.

We define a distance in the space BS as follows [4]: For any $\phi, \psi \in BS$, we define

$$\rho(\phi, \psi) = \Sigma \frac{\rho_j(\phi, \psi)}{2^j[1 + \rho_j(\phi, \psi)]},$$

where

$$\rho_j(\phi, \psi) = \sup_{-j \le s \le 0} |\phi(s) - \psi(s)|.$$

It is clear that $\rho(\phi^k, \phi) \to 0$ as $k \to \infty$ if and only if $\phi^k(s) \to \phi(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $k \to \infty$ [4].

Definition 1.1. The bounded solution $u(n)$ of (1.2) is called *BC-uniformly stable* (in short, BC-US) if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \ge 0$, $\phi \in BS$ with $|\phi - u_{n_0}|_{BC} < \delta$, then

$$|x(n) - u(n)| < \varepsilon, n \ge n_0,$$

where $x(n)$ denotes any solution of (1.2) through (n_0, ϕ).

Definition 1.2. The bounded solution $u(n)$ of (1.2) is called *(K, ρ)-uniformly stable* ((K, ρ)-US) if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \ge 0$, $\phi \in BS$ with $\phi(s) \in K$, $s \le 0$, and $\rho(\phi, u_{n_0}) < \delta$, then

$$\rho(x_n, u_n) < \varepsilon, \ n \ge n_0,$$

where $x(n)$ denotes any solution of (1.2) through (n_0, ϕ).

Definition 1.3. The bounded solution $u(n)$ of (1.2) is called *$((K, \rho), \mathbb{R}^d)$-uniformly stable* ($((K, \rho), \mathbb{R}^d)$-US) if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \ge 0$, $\phi \in BS$ with $\phi(s) \in K$, $s \le 0$ and $|\phi(s) - u_{n_0}(s)| < \delta$, then

$$|x(n) - u(n)| < \varepsilon, \ n \ge n_0,$$

where $x(n)$ denotes any solution of (1.2) through (n_0, ϕ).

Definition 1.4. The bounded solution $u(n)$ of (1.2) is said to be *BC-uniformly asymptotically stable* (BC-UAS) if it is BC-US and if there exists a $\delta_0 > 0$ such that for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) > 0$ for which

$$|x(n) - u(n)| < \varepsilon, \ n \ge n_0 + N$$

whenever $n_0 \ge 0$, $\phi \in BS$ and $|\phi - u_{n_0}|_{BS} < \delta_0$.

Definition 1.5. The bounded solution $u(n)$ of (1.2) is said to be (K,ρ)-*uniformly asymptotically stable* ((K,ρ)-UAS) if it is (K,ρ)-US and if there exists a $\delta_0 > 0$ such that for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) > 0$ for which

$$\rho(x_n, u_n) < \varepsilon, n \geq n_0 + N$$

whenever $n_0 \geq 0$, $\phi \in BS$ with $\phi(s) \in K$, $s \leq 0$, and $\rho(\phi, u_{n_0}) < \delta_0$.

Definition 1.6. The bounded solution $u(n)$ of (1.2) is said to be $((K,\rho),\mathbb{R}^d)$-*uniformly asymptotically stable* ($((K,\rho),\mathbb{R}^d)$-UAS) if it is $((K,\rho),\mathbb{R}^d)$-US and if there exists a $\delta_0 > 0$ such that for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) > 0$ for which

$$|x(n) - u(n)| < \varepsilon, n \geq n_0 + N$$

whenever $n_0 \geq 0$, $\phi \in BS$ with $\phi(s) \in K$, $s \leq 0$, and $|\phi(s) - u_{n_0}(s)| < \delta_0$.

Definition 1.7. The bounded solution $u(n)$ of (1.2) is called *BS-totally stable* (BS-TS) if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \geq 0$, $|x_{n_0} - u_{n_0}|_{BS} < \delta$ and $h \in BS([n_0,\infty))$ with $|h|_{[n_0,\infty)} < \delta$, then

$$|x(n) - u(n)| < \varepsilon, n \geq n_0,$$

where $x(n)$ is any solution of $x(n+1) = f(n, x_n) + h(n)$, $n \geq 0$, through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s)$, $s \leq 0$.

Definition 1.8. The bounded solution $u(n)$ of (1.2) is called (K,ρ)-*totally stable* ((K,ρ)-TS) if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta$ and $h \in \mathrm{BS}([n_0,\infty))$ with $|h|_{[n_0,\infty)} < \delta$, then

$$\rho(x_n, u_n) < \varepsilon, n \geq n_0,$$

where $x(n)$ is any solution of $x(n+1) = f(n, x_n) + h(n)$, $n \geq 0$, through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s)$, $s \leq 0$.

Definition 1.9. The bounded solution $u(n)$ of (1.2) is called $((K,\rho),\mathbb{R}^d)$-*totally stable* ($((K,\rho),\mathbb{R}^d)$-TS) if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \geq 0$, $|x(n_0) - u(n_0)| < \delta$ and $h \in \mathrm{BS}([n_0,\infty))$ with $|h|_{[n_0,\infty)} < \delta$, then

$$|x(n) - u(n)| < \varepsilon, n \geq n_0,$$

where $x(n)$ is any solution of $x(n+1) = f(n, x_n) + h(n)$, $n \geq 0$, through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s)$, $s \leq 0$.

REMARK 1.10. ρ-stability implies BS-stability because of

$$\rho(\phi, \psi) \leq |\phi - \psi|_{BS}, \ \phi, \psi \in BS.$$

Also, (K, ρ)-stability implies $((K, \rho), \mathbb{R}^d)$-stability.

2. Main results

We consider the functional difference equation

$$x(n+1) = f(n, x_n), \ n \in \mathbb{Z}^+ \tag{2.1}$$

with the following assumptions:

(H1) $\sup\{f(n, \phi) : n \in \mathbb{Z}^+, \phi \in B \text{ with } |\phi|_B \leq H\} = L_0(H) < \infty$ for any $H > 0$.

(H2) $f : \mathbb{Z}^+ \times B \to \mathbb{R}^d$ is uniformly continuous at second variable $\phi \in S$ for any compact set S in B and is almost periodic in n uniformly for $\phi \in B$, i.e., for any $\varepsilon > 0$ and any compact set S in B, there exists a positive integer $L^*(\varepsilon, S)$ such that any interval of length L^* contains an integer τ for which

$$|f(n+\tau, \phi) - f(n, \phi)| < \varepsilon, \ n \in \mathbb{Z}, \phi \in S.$$

(H3) Eq. (2.1) has a bounded solution $u(n)$ defined on $\mathbb{Z}^+$ which passes through $(0, u_0)$, i.e., $\sup_{n \in \mathbb{Z}^+} |u(n)| < \infty$ and $u_0 \in BS$.

We denote by $x(\cdot, \sigma, \phi)$ the solution of (2.1) through (σ, ϕ).

The concept of normality of almost periodic functions is equivalent to the above definition: for any sequence $(h_k{}') \subset \mathbb{Z}$, there exist a subsequence $(h_k) \subset (h_k{}')$ and a function $g(n, \phi)$ such that

$$f(n + h_k, \phi) \to g(n, \phi)$$

uniformly on $\mathbb{Z} \times S$ as $k \to \infty$, where S is any compact set in B. We denote by $H(f)$ the set of all limit functions g such that for some sequence $(n_k) \subset \mathbb{Z}$ with $n_k \to \infty$ as $k \to \infty$ and

$$f(n + n_k, \phi) \to g(n, \phi)$$

uniformly on $\mathbb{Z} \times S$ as $k \to \infty$.

THEOREM 2.1. *Assume that (H3). Then the bounded solution $u(n)$ of (2.1) is (K, ρ)-US if and only if $((K, \rho), \mathbb{R}^d)$-US.*

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) Let $\varepsilon > 0, n \geq 0, \phi \in BS$ with $\phi(s) \in K, s \leq 0$.

Suppose that $\rho(\phi, u_{n_0}) < \delta$. Since $u(n)$ is $((K,\rho), \mathbb{R}^d)$-US, we have

$$|x(n) - u(n)| < \varepsilon,\ n \geq n_0,$$

where $x(n)$ is any solution of (2.1) through (n_0, ϕ). Then

$$\sup\{|x(n+s+n_0) - u(n+s+n_0)| : n_0 \geq 0, -j \leq s \leq n_0, \\ \rho(\phi, u_{n_0}) < \delta \text{ with } \phi(\cdot) \in K\} \to 0 \text{ as } n \to \infty \text{ for each } j > 0.$$

If follows that

$$\sup\{\rho(x_{n+n_0}, u_{n+n_0}) : n_0 \geq 0, \rho(\phi, u_{n_0}) < \delta \text{ with } \phi(\cdot) \in K\} \\ \to 0 \text{ as } n \to \infty.$$

Hence we obtain

$$\rho(x_n, u_n) < \varepsilon,\ n \geq n_0.$$

□

By the same manner of Theorem 2.1, we obtain the following.

THEOREM 2.2. *Assume that (H3). Then the bounded solution $u(n)$ of (2.1) is (K,ρ)-UAS if and only if $((K,\rho),\mathbb{R}^d)$-UAS.*

THEOREM 2.3. *Suppose that (H1), H(2), and (H3). If the bounded solution $u(n)$ of (2.1) is BS-UAS if and only if BS-TS.*

Proof. Let $\varepsilon > 0, n \geq 0, h \in BS([n_0,\infty))$ with $|h|_{[n_0,\infty)} \leq \delta(\varepsilon)$, and $|x_{n_0} - u_{n_0}| < \delta(\varepsilon)$, where $(\delta(\cdot), \delta_0, N(\cdot))$ is the triple for BC-UAS of $u(n)$ and $x(n)$ is any solution of $x(n+1) = f(n, x_n) + h(n), n \geq 0$, through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s), s \leq 0$. Assume that $\delta_0 < \delta(1)$.

We claim that if $(v, g) \in H(u, f)$ and

$$|\phi - v_{n_0}|_{BS} < \delta_0,$$

then

$$|y(n) - v(n)| < \varepsilon,\ n \geq n_0 + N(\frac{\varepsilon}{2}), \tag{2.2}$$

where $y(\cdot)$ is any solution of $x(n+1) = g(n, x_n)$ through (n_0, ϕ). From $(v, g) \in H(u, f)$, there exists a sequence $(\tau_k) \subset \mathbb{Z}^+$ with $\tau_k \to \infty$ as $k \to \infty$ such that

$$(u_{\tau_k}, f_{\tau_k}) \to (v, g)$$

uniformly on any compact set in $\mathbb{Z}^+ \times B$. Consider any solution $x(\cdot, n_0 + \tau_k, \phi - v_{n_0} + u_{n_0+\tau_k})$ of (2.1). For any $n \in \mathbb{N}$, we set

$$x^k(n) = x(n + \tau_k, n_0 + \tau_k, \phi - v_{n_0} + u_{n_0+\tau_k}),\ n \in \mathbb{Z}.$$

Then

$$|(x^k)_{n_0} - u_{n_0+\tau_k}|_{BS} = |\phi - v_{n_0}|_{BS} < \delta(\frac{\eta}{2}).$$

Since $u(n)$ is BS-US, we have

$$|x^k(n) - u(n + \tau_k)|_{BS} < \frac{\eta}{2},\ n \geq n_0, k \in \mathbb{N}. \tag{2.3}$$

Thus

$$\sup\{|(x^k)_n|_B : n \geq n_0, k \in \mathbb{N}\} \leq M(\frac{\eta}{2} + |u|_{[n_0,\infty)}) + N|\phi - v_{n_0} + u_{n_0+\tau_k}|$$

by Axiom (A1). Hence there exists a function $y : [n_0, \infty) \to \mathbb{R}^d$ such that

$$x^k(n) \to y(n)$$

uniformly on $[n_0, \infty)$ as $k \to \infty$. Note that

$$x^k(n_0) = \phi(0) - v(n_0) + u(n_0 + \tau_k).$$

Since $u(n_0 + \tau_k) \to v(n_0)$ as $k \to \infty$, we obtain

$$y(n_0) = \phi(0).$$

Extend y by setting $y_{n_0} = \phi$. Then $y \in BS(\mathbb{Z}^-, \mathbb{R}^d)$ and

$$|(x^k)_n - y_n|_B \text{ or } x^k(n) \to y(n)$$

uniformly on any compact set in $[n_0, \infty)$ as $k \to \infty$.

If we repeat the above argument, then we obtain

$$|y(n) - v(n)| < \varepsilon,\ n \geq n_0 + N(\frac{\eta}{2}) \tag{2.4}$$

whenever $(v, g) \in H(u, f)$ and $|\phi - v_{n_0}|_{BS} < \delta_0$.

Now, we assume that $u(n)$ is not BS-TS. Then there exist an ε with $0 < \varepsilon < \delta_0$, sequences $(\tau_k) \subset \mathbb{Z}^-, (r_k) \subset \mathbb{Z}^-, (\phi_k) \subset BS$, $(h_k) \subset BS([\tau_k, \infty))$, and solutions $z^k(n) = x(\cdot, \tau_k, \phi^k)$ such that

$$|\phi^k - u_{\tau_k}| < \frac{1}{k}, \tag{2.5}$$

$$|h_k|_{[\tau_k,\infty)} < \frac{1}{k},\ k \in \mathbb{N}$$

$$|z^k(\tau_k + r_k) - u(\tau_k + r_k)| = \varepsilon$$

$$|z^k(n) - u(n)| < \varepsilon,\ -\infty < n < \tau_k + r_k. \tag{2.6}$$

By the same method as in the proof of [6], we can show that (r_k) is bounded. Thus we assume that $r_k \to r$ for some r with $0 \le r < \infty$, as $k \to \infty$. Also, we may assume that $z^k(\tau_k + n) \to \xi$ for some function ξ, uniformly on any compact set in $(-\infty, r]$. In this case, (τ_k) is also bounded. Hence $\tau_k \to \tau$ for some τ as $k \to \infty$. Then $\xi(n-\tau)$ is a solution of (2.1) on $[\tau, \tau + r]$. Moreover, we have

$$|\xi_0 - u_\tau|_{BS} = 0$$

and

$$|\xi(r) - u(\tau + r)| = \varepsilon$$

by (2.5) and (2.6). This contradicts the fact that $u(n)$ is BS-US. This completes the proof. □

THEOREM 2.4. *Suppose that (H1), H(2), and (H3). Moreover, assume that the solution $v(n)$ of the limiting equation $x(n+1) = g(n, x_n)$, $(v, g) \in H(u, f)$ is unique for the initial conditions. Then the solution of (2.1) is BS-UAS if and only if (K, ρ)-UAS.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Let $(\delta(\cdot), \delta_0, N(\cdot))$ be the triple for BS-UAS of $u(n)$. Assume $\delta_0 < \delta(1) < 1$. In view of Theorem 2.3, $u(n)$ is BS-TS. Then $u(n)$ is (K, ρ)-TS by Theorem 3 in [4]. Thus we show that for any $\varepsilon > 0$ there exists an $\overline{N}(\varepsilon) > 0$ such that

$$\rho(\phi, u_{n_0}) < \delta_1 = \overline{\delta}(\frac{\delta_0}{4})$$

with $\phi(\cdot) \in K$ implies

$$\rho(x_n(n_0, \phi), u_n) < \varepsilon,\ n \ge n_0 + \overline{N}(\varepsilon),$$

where $\overline{\delta}(\cdot)$ is the number for the (K, ρ)-TS of $u(n)$. If this is not the case, then there exist an $\varepsilon > 0$ and sequences $(\tau_k) \subset \mathbb{Z}^+, (n_k) \subset \mathbb{Z}^+$ with $n_k \ge \tau_k, (\phi^k) \subset BS$, and solutions $x(\cdot, \tau_k, \phi^k)$ such that

$$\rho(\phi^k, u_{\tau_k}) < \delta_1,\ \phi^k(\cdot) \in K \tag{2.7}$$

and

$$\rho(x_{n_k}(\tau_k, \phi^k), u_{n_k}) \ge \varepsilon, k \in \mathbb{N}. \tag{2.8}$$

Thus, from (2.7) and (2.8), we get

$$\rho(x_{n_k}(\tau_k, \phi^k), u_n) \ge \frac{\delta_0}{4}, n \ge \tau_k \tag{2.9}$$

and

$$\rho(x_{n+\tau_k+k}(\tau_k, \phi^k), u_{n+\tau_k+k}) \ge \overline{\delta}(\varepsilon), -k \le n \le k. \tag{2.10}$$

We set $x^k(n) = x(n + \tau_k + k, \tau_k, \phi^k),\ n \in \mathbb{Z}$. Note that

$$\begin{aligned}\rho(\phi,\psi) &\geq \frac{|\phi-\psi|}{2[1+|\phi-\psi|]}\\ &\geq \frac{|\phi(0)-\psi(0)|}{2[1+|\phi(0)-\psi(0)|]}.\end{aligned}$$

Then, if $\rho(\phi,\psi) \leq \frac{1}{2}$, then

$$|\phi(0)-\psi(0)| \leq \frac{2\rho(\phi,\psi)}{1-2\rho(\phi,\psi)}.$$

Hence we have

$$|x^k(n) - u(n+\tau_k+k)| \leq \frac{\delta_0}{2-\delta_0},\ -k \leq n \leq k, \tag{2.11}$$

by (2.9). We assume that there exists a bounded function $\mu \in BS$ such that $x^k(n) \to \mu(n)$ uniformly for any compact set in $\mathbb{Z}$. Also, there exists $(v, g) \in H(u, f)$ such that

$$(u_{\tau_k+k}, f_{\tau_k+k}) \to (v, g)$$

uniformly for any compact set in $\mathbb{Z}^+ \times B$. Then $\mu(n) = y(n, 0, \mu_0)$ on $\mathbb{Z}$. By letting $k \to \infty$ in (2.11), we obtain

$$|\mu(n) - v(n)| \leq \frac{\delta_0}{2-\delta_0},\ n \in \mathbb{Z}.$$

In particular,

$$|\mu_0 - v_0|_{BS} \leq \frac{\delta_0}{2-\delta_0} < \delta_0.$$

Therefore, by (2.4), $|\mu(n) - v(n)| \to 0$ as $n \to \infty$. On the other hand, we have

$$\rho(\mu_n, v_n) \geq \overline{\delta}(\varepsilon),\ n \in \mathbb{Z}$$

by letting $k \to \infty$ in (2.10), which is a contradiction. Consequently, $u(n)$ is (K,ρ)-UAS. This completes the proof. □

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